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q-Analogues for Green functions for powers of the invariant Laplacian in the unit disc

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1 Introduction

In 1993 W.K. Hayman and B. Korenblum published among other results explicit formulae for Green functions of powers of the Laplace operator in the balls in $\mathbb{R}^n$ (see [4]). J. Peetre and M. Engliš [2] have obtained analogous results for some powers of the Möbius-invariant Laplace operator in the unit ball in $\mathbb{C}^n$. In the particular case of the unit disc $\mathbb{U} \subset \mathbb{C}$ they have presented explicit formulae for Green functions of the powers Δ, Δ^2, Δ^3, Δ^4 of the Möbius-invariant (equivalently, $SU(1,1)$-invariant) Laplace operator. For Δ and Δ^2 the Green functions are

$$\frac{1}{4\pi}\ln t,$$

$$\frac{1}{16\pi}(-\ln t \ln(1-t) - 2\mathrm{Li}_2(t) + \frac{\pi^2}{3}),$$

where

$$t = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2},$$

$z, w \in \mathbb{U}$, $\mathrm{Li}_2(t) = \sum_{m=1}^{\infty} \frac{t^m}{m^2}$ is Euler's dilogarithm.

The aim of the present work is the computation of q-analogues for these kernels. Namely, we will concern with the quantum unit disc which is a homogeneous space of the quantum group $SU(1,1)$, and q is the parameter used in the theory of quantum groups [1]. Of course, all our formulae become the classical ones at the limit $q \to 1$.

The statement of our main result (Theorem 2.2) is: for any finite function f in the quantum unit disc the following equalities hold

$$\Delta_q^{-1} f = \int\limits_{\mathbb{U}_q} \mathbb{G}_1 f \, d\nu, \qquad (1.1)$$

$$\Delta_q^{-2} f = \int\limits_{\mathbb{U}_q} \mathbb{G}_2 f \, d\nu, \qquad (1.2)$$

(here Δ_q and $\int\limits_{\mathbb{U}_q} \cdot d\nu$ are q-analogues of the $SU(1,1)$-invariant Laplace operator and invariant integral in the unit disc respectively, $\mathbb{G}_1$ and $\mathbb{G}_2$ are certain kernels given by explicit formulae (2.17),(2.18); precise definitions are to be found below).

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Let us outline ideas of the paper. In the classical case all the integral operators we are interested in are intertwining (i.e., they commute with the action of the group $SU(1,1)$ in spaces of functions in the unit disc). Thus the kernels of these operators are functions in the simplest one:

$$\frac{(1-z\bar{z})(1-\zeta\bar{\zeta})}{(1-z\bar{\zeta})(1-\zeta\bar{z})}. \tag{1.3}$$

In [8] an algebra was considered of kernels of integral operators in the quantum disc, and q-analogues were obtained of integer negative powers of (1.3):

$$G_{-l}=\{(1-\zeta\zeta^*)^{-l}(q^2z^*\zeta;q^2)_l(z\zeta^*;q^2)_l(1-z^*z)^{-l}\}, \tag{1.4}$$

with $(a;q)_l \stackrel{\mathrm{def}}{=} (1-a)\cdot(1-aq)\cdot\ldots\cdot(1-aq^{l-1})$. Furthermore, the kernels G_l may be defined for any $l\in\mathbb{C}$ by "analytic continuation" in the parameter q^{-2l} (see [8] or Section 2 of the present paper).

In the classical case "any" function of the kernel (1.3) can be expanded in integral by powers of this kernel using the Melline transform. It turned out that in the quantum case as well as in the classical one the kernels $\mathbb{G}_1$, $\mathbb{G}_2$ of the **intertwining** integral operators Δ_q^{-1}, Δ_q^{-2} can be written in the form

$$\mathbb{G}_1=\int G_l d\sigma_1(l),$$

$$\mathbb{G}_2=\int G_l d\sigma_2(l),$$

with some distributions $d\sigma_1(l)$, $d\sigma_2(l)$. We find these integrals applying the operators Δ_q^{-1}, Δ_q^{-2} to a q-analogue (denoted by f_0 in the text) of the delta-function at the centre of the disc.

2 Statement of results

First of all, we remind some notations and results on function theory in the quantum unit disc (see [6]-[9]).

Let $q\in(0;1)$. We impose the notation $\mathrm{Pol}(\mathbb{C})_q$ for the involutive algebra given by its generator z and the unique commutation relation

$$z^*z=q^2zz^*+1-q^2. \tag{2.1}$$

Let $y\stackrel{\mathrm{def}}{=}1-zz^*$. It is straightforward that $y=y^*$,

$$z^*y=q^2yz^*, \qquad zy=q^{-2}yz, \tag{2.2}$$

and any element $f\in\mathrm{Pol}(\mathbb{C})_q$ admits a unique decomposition

$$f=\sum_{m>0}z^m\psi_m(y)+\psi_0(y)+\sum_{m>0}\psi_{-m}(y)z^{*m}. \tag{2.3}$$

It is also not hard to show that the algebra $\mathrm{Pol}(\mathbb{C})_q$ admits a unique up to unitary equivalence **faithful** $*$-representation by bounded operators in a Hilbert space and the spectrum of the operator, corresponding to the element y, is the set $\{0\}\bigcup q^{2\mathbb{Z}_+}$ (we shall use notation y both for the element of the polynomial algebra as well as for an indeterminate in the set $q^{2\mathbb{Z}_+}$). This allows one to introduce the algebra $D(\mathbb{U})_q$ of finite functions in the quantum unit disc. By the definition it consists of **finite** series of the form (2.3) with $\mathrm{supp}\psi_m \subset q^{2\mathbb{Z}_+}$, $\mathrm{card}(\mathrm{supp}\psi_m) < \infty$.

The linear functional [7, Theorem 3.5]

$$\int_{\mathbb{U}_q} f d\nu \stackrel{\text{def}}{=} (1-q^2)\sum_{m=o}^{\infty}\psi_0(q^{2m})q^{-2m}, \tag{2.4}$$

where $f = \sum_{m>0} z^m\psi_m(y) + \psi_0(y) + \sum_{m>0}\psi_{-m}(y)z^{*m} \in D(\mathbb{U})_q$, is a q-analogue for the $SU(1,1)$-invariant integral in the unit disc (see Section 3). We impose the notation $L^2(d\nu)_q$ for the completion of $D(\mathbb{U})_q$ with respect to the norm

$$\|f\| \stackrel{\text{def}}{=} \left(\int_{\mathbb{U}_q} f^* f d\nu\right)^{\frac{1}{2}}. \tag{2.5}$$

We need the well known first-order differential calculus over $\mathrm{Pol}(\mathbb{C})_q$. It is a $\mathrm{Pol}(\mathbb{C})_q$-module $\Omega^1(\mathbb{C})_q$ given by its generators dz, dz^*, and the commutation relations

$$z\cdot dz = q^{-2}dz\cdot z,\ \ z^*\cdot dz^* = q^2 dz^*\cdot z^*,\ \ z^*\cdot dz = q^2 dz\cdot z^*,\ \ z\cdot dz^* = q^{-2}dz^*\cdot z, \tag{2.6}$$

and equipped with a linear map $d: \mathrm{Pol}(\mathbb{C})_q \to \Omega^1(\mathbb{C})_q$ such that

1. $d: z\mapsto dz, \quad z^*\mapsto dz^*$;
2. $d(f_1 f_2) = df_1\cdot f_2 + f_1\cdot df_2$ for any $f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q$ (the Leibniz rule).

The partial derivatives $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial z^*}$ in $\mathrm{Pol}(\mathbb{C})_q$ are given by

$$df = dz\frac{\partial f}{\partial z} + dz^*\frac{\partial f}{\partial z^*}.$$

The operator

$$\Delta_q f \stackrel{\text{def}}{=} (1-zz^*)^2\frac{\partial}{\partial z^*}\frac{\partial}{\partial z}f \tag{2.7}$$

is a q-analogue of the invariant Laplacian in the unit disc (see [7, 9]).

One can define the operators $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial z^*}$, Δ_q on the space of finite functions (it is sufficient to use the formulae

$$df(y) = -q^2\frac{f(y)-f(q^2y)}{y-yq^2}z^*dz - z\frac{f(y)-f(q^2y)}{y-yq^2}dz^*,$$

$$f(y)dz = dzf(y),$$

$$f(y)dz^* = dz^*f(y),$$

for $f\in \mathrm{Pol}(\mathbb{C})_q$).

The following result was announced in [6, Proposition 3.2]:

Theorem 2.1 *The operator Δ_q can be extended to the selfadjoint bounded invertible operator in $L^2(d\nu)_q$.*

To formulate our results we need the notion of an integral operator in the quantum case.

Impose the notation $D(\mathbb{U}\times\mathbb{U})'_q$ for the space of formal series of the form

$$f=\sum_{(i,j)\in\mathbb{Z}^2} f_{ij}, \tag{2.8}$$

$$f_{ij}=\begin{cases} z^i\otimes 1\cdot\psi_{ij}(y\otimes 1,1\otimes y)\cdot 1\otimes z^j, & i\geq 0, j\geq 0,\\ z^i\otimes 1\cdot\psi_{ij}(y\otimes 1,1\otimes y)\cdot 1\otimes z^{*j}, & i\geq 0, j<0,\\ z^{*i}\otimes 1\cdot\psi_{ij}(y\otimes 1,1\otimes y)\cdot 1\otimes z^j, & i<0, j\geq 0,\\ z^{*i}\otimes 1\cdot\psi_{ij}(y\otimes 1,1\otimes y)\cdot 1\otimes z^{*j}, & i<0, j<0,\end{cases} \tag{2.9}$$

with $\{\psi_{ij}\}$ being any functions on $q^{2\mathbb{Z}_+}\times q^{2\mathbb{Z}_+}$. It is convenient in the sequel to write $z,y,z^*,\zeta,\eta,\zeta^*$ instead of $z\otimes 1, y\otimes 1, z^*\otimes 1, 1\otimes z, 1\otimes y, 1\otimes z^*$, respectively. Note that $D(\mathbb{U}\times\mathbb{U})'_q$ can be made into a topological vector space. We describe the topology in Section 3.

Let $D(\mathbb{U})'_q$ be the space of formal series of the form (2.3) with $\{\psi_m(y)\}_{m\in\mathbb{Z}}$ being any functions on $q^{2\mathbb{Z}_+}$. Then one shows that for any $K\in D(\mathbb{U}\times\mathbb{U})'_q$ the map

$$\int\limits_{\mathbb{U}_q} Kf d\nu \stackrel{\text{def}}{=} id\otimes\nu\,(K\cdot 1\otimes f)\,,\quad f\in D(\mathbb{U})_q,\quad \nu(f)\stackrel{\text{def}}{=}\int\limits_{\mathbb{U}_q} f d\nu\,,$$

is a well defined operator from $D(\mathbb{U})_q$ into $D(\mathbb{U})'_q$. (Indeed, it follows from the relations

$$z^*\phi(y)=\phi(q^2y)z^*,\qquad z\phi(y)=\phi(q^{-2}y)z,\qquad \text{supp}\phi\subset q^{2\mathbb{Z}_+}$$

and

$$z^{*k}z^k=(1-q^2y)(1-q^4y)\dots(1-q^{2k}y),\ z^kz^{*k}=(1-y)(1-q^{-2}y)\dots(1-q^{-2k+2}y),$$

that $D(\mathbb{U})'_q$ is a $D(\mathbb{U})_q$-bimodule. Now to prove the correctness of the definition it sufficies to observe that for a finite $f_2(y)$

$$\int\limits_{\mathbb{U}_q} z^kf_1(y)z^jf_2(y)d\nu=0,\quad k\neq 0\quad\text{or}\quad j\neq 0,$$

$$\int\limits_{\mathbb{U}_q} f_1(y)z^{*k}z^jf_2(y)d\nu=0,\quad k\neq j,$$

$$\int\limits_{\mathbb{U}_q} f_1(y)z^{*k}f_2(y)z^{*j}d\nu=0,\quad k\neq 0\quad\text{or}\quad j\neq 0,$$

$$\int\limits_{\mathbb{U}_q} z^kf_1(y)f_2(y)z^{*j}d\nu=0,\quad k\neq j.)$$

Such operators can be treated as integral operators.

We have already mentioned (see (1.4)) one important set of kernels introduced in [8, Section 6]:

$$G_{-l}=\{(1-\zeta\zeta^*)^{-l}(q^2z^*\zeta;q^2)_l(z\zeta^*;q^2)_l(1-z^*z)^{-l}\},\qquad l=1,2,3\ldots, \tag{2.10}$$

where $(a;q)_n\overset{\rm def}{=}(1-a)(1-aq)\dots(1-aq^{n-1})$, and the brackets $\{\}$ indicate that z,z^* should be multiplied within the algebra $\mathrm{Pol}(\mathbb{C})_q^{op}$ derived from $\mathrm{Pol}(\mathbb{C})_q$ by replacing its product to the opposite one (for example, $\{z\cdot z^*\}=z^*\cdot z$). These kernels are invariant in a sense (we explain the term "invariant" in Section 3; see also [8]) and therefore may be regarded as q-analogues of the kernels

$$\left(\frac{(1-z\bar{z})(1-\zeta\bar{\zeta})}{(1-z\bar{\zeta})(1-\zeta\bar{z})}\right)^{-l}, \tag{2.11}$$

with l being a positive integer number. Using relation (1.3.2) from [3] one can rewrite

$$G_{-l}=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}q^{2k}\frac{(q^{-2l};q^2)_k\cdot(q^{-2l};q^2)_n}{(q^2;q^2)_k\cdot(q^2;q^2)_n}z^ny^{-l}z^{*k}\zeta^k\eta^{-l}\zeta^{*n}, \tag{2.12}$$

(the sum is finite). It is evident that for any finite functions ϕ_1,ϕ_2 the function $t\overset{\rm def}{=}q^{-2l}\mapsto g_{\phi_1,\phi_2}(t)\overset{\rm def}{=}\int\limits_{\mathbb{U}_q}\phi_1\cdot\left(\int\limits_{\mathbb{U}_q}G_{-l}\cdot\phi_2 d\nu\right)d\nu$ belongs to $\mathbb{C}[t,t^{-1}]$. This observation allows one to prove (just as it was done in [8]) that there exists a unique vector-function of a **complex** variable t with values in $D(\mathbb{U}\times\mathbb{U})_q'$ which coincides with the right side of (2.12) for $t\in q^{-2\mathbb{N}}$. In the sequel G_l will stand for this "analytic continuation". In this way one obtains q-analogues of (2.11) for any complex power l.

We will need also the kernels

$$\hat{G}_N\overset{\rm def}{=}\lim_{l\to N}\frac{G_l-G_N}{l-N},\qquad N=1,2,\dots \tag{2.13}$$

(the limit in the topology mentioned above).

Remark. Let

$$L_0(\xi)=0,\quad L_k(\xi)\overset{\rm def}{=}\frac{1}{1-\xi}+\frac{q^2}{1-q^2\xi}+\dots+\frac{q^{2k-2}}{1-q^{2k-2}\xi},\quad k=1,2,\dots,\infty.$$

Using (2.12) and formulae

$$\frac{d}{dl}y^l=\ln y\cdot y^l,\quad \frac{d}{dl}(q^{2l};q^2)_n=h\cdot L_n(q^{2l})\cdot(q^{2l};q^2)_n\,,$$

one can show that for $h\overset{\rm def}{=}\ln q^{-2}$

$$\hat{G}_N=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}q^{2k}\frac{(q^{2N};q^2)_k\cdot(q^{2N};q^2)_n}{(q^2;q^2)_k\cdot(q^2;q^2)_n}\cdot\Psi_{N,k,n}, \tag{2.14}$$

where

$$\Psi_{N,k,n} \stackrel{\text{def}}{=} h(L_k(q^{2N}) + L_n(q^{2N}))z^n y^N z^{*k}\zeta^k\eta^N\zeta^{*n} + z^n y^N \ln(y) z^{*k}\zeta^k\eta^N\zeta^{*n}$$
$$+z^n y^N z^{*k}\zeta^k\eta^N \ln(\eta)\zeta^{*n}.$$

Note that

$$\lim_{q\to 1} G_l = \left(\frac{(1-z\bar{z})(1-\zeta\bar{\zeta})}{(1-z\bar{\zeta})(1-\zeta\bar{z})}\right)^l,$$

$$\lim_{q\to 1} \hat{G}_N = \left(\frac{(1-z\bar{z})(1-\zeta\bar{\zeta})}{(1-z\bar{\zeta})(1-\zeta\bar{z})}\right)^N \cdot \ln\left(\frac{(1-z\bar{z})(1-\zeta\bar{\zeta})}{(1-z\bar{\zeta})(1-\zeta\bar{z})}\right).$$

The principal result of the present work is

Theorem 2.2 *For any $f \in D(\mathbb{U})_q$*

$$\Delta_q^{-1} f = \int\limits_{\mathbb{U}_q} \mathbb{G}_1 f d\nu, \tag{2.15}$$

$$\Delta_q^{-2} f = \int\limits_{\mathbb{U}_q} \mathbb{G}_2 f d\nu, \tag{2.16}$$

where

$$\mathbb{G}_1 \stackrel{\text{def}}{=} -\sum_{m=1}^{\infty} \frac{q^{-2}-1}{q^{-2m}-1} G_m\,, \tag{2.17}$$

$$\mathbb{G}_2 \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \frac{q^{2m-2}(1+q^{2m})(1-q^2)^2}{(1-q^{2m})^2} G_m - \frac{1-q^2}{h} \sum_{m=1}^{\infty} \frac{q^{-2}-1}{q^{-2m}-1} \hat{G}_m\,, \tag{2.18}$$

and $G_m, \hat{G}_m$ are given by (2.12), (2.13), respectively.

3 Auxiliary result: radial part of the invariant Laplacian

It is easy to check by direct calculations that for any $f(y) \in \mathrm{Pol}(\mathbb{C})_q$ or $D(\mathbb{U})_q$

$$\Delta_q f(y) = q^{-1} y^2 D(1-qy) D f(y),$$

where

$$(Df)(t) \stackrel{\text{def}}{=} \frac{f(q^{-1}t) - f(qt)}{q^{-1}t - qt}.$$

Let $L^2(d\nu)_q^{(0)} \stackrel{\text{def}}{=} \{f(y) \in D(\mathbb{U})'_q | \sum_{m=0}^{\infty} |f(q^{2m})|^2 q^{-2m} < \infty\}$. The following proposition is proved in [7, Lemma 5.5]:

Proposition 3.1 $\Delta_q^{(0)} \stackrel{\text{def}}{=} q^{-1}y^2 D(1-qy)D$ *is a bounded selfadjoint invertible operator in $L^2(d\nu)_q^{(0)}$.*

The term "radial part of the invariant Laplacian in the quantum disc" stand for this operator.

Let $f_0 = f_0(y)$ be such a finite function that

$$f_0 = \begin{cases} 1 & y = 1, \\ 0 & y = q^{2k}, k = 1, 2, \ldots . \end{cases} \tag{3.1}$$

In this section we will prove the following

Theorem 3.2

$$(\Delta_q^{(0)})^{-1} f_0 = g_1(y),$$
$$(\Delta_q^{(0)})^{-2} f_0 = g_2(y),$$

where

$$g_1(y) = -(1 - q^2) \sum_{m=1}^{\infty} \frac{q^{-2} - 1}{q^{-2m} - 1} y^m, \tag{3.2}$$

$$g_2(y) = (1 - q^2)$$

$$\times \left(\sum_{m=1}^{\infty} \frac{q^{2m-2}(1 + q^{2m})(1 - q^2)^2}{(1 - q^{2m})^2} y^m - \frac{1 - q^2}{h} \ln y \sum_{m=1}^{\infty} \frac{q^{-2} - 1}{q^{-2m} - 1} y^m \right). \tag{3.3}$$

Remind some well known notations [3]:

$$(a; q)_n \stackrel{\text{def}}{=} (1 - a)(1 - aq) \ldots (1 - aq^{n-1}),$$

$$(a; q)_\infty \stackrel{\text{def}}{=} (1 - a)(1 - aq) \ldots (1 - aq^{n-1}) \ldots ,$$

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x},$$

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, \ldots , a_r; q; z \\ b_1, \ldots , b_s \end{matrix} \right]$$

$$= \sum_{n \in \mathbb{Z}_+} \frac{(a_1; q)_n \cdot (a_2; q)_n \cdot \ldots \cdot (a_r; q)_n}{(b_1; q)_n \cdot (b_2; q)_n \cdot \ldots \cdot (b_s; q)_n (q; q)_n} \left((-1)^n \cdot q^{\frac{n(n-1)}{2}} \right)^{1+s-r} \cdot z^n,$$

$$\int_0^1 f(y) d_{q^2} y = (1 - q^2) \sum_{m=0}^{\infty} f(q^{2m}) q^{2m}.$$

We will use the following results from [6, 7, 9]:

Proposition 3.3 *The functions*

$$\varphi_\rho(y) \stackrel{\text{def}}{=} {}_3\Phi_2 \left[\begin{matrix} y^{-1}, q^{1+2i\rho}, q^{1-2i\rho}; q^2; q^2 \\ q^2, 0 \end{matrix} \right], \tag{3.4}$$

$$\psi_\rho(y) \stackrel{\text{def}}{=} y^{\frac{1}{2} - i\rho} \cdot {}_2\Phi_1 \left[\begin{matrix} q^{1-2i\rho}, q^{1-2i\rho}; q^2; q^2 y \\ q^{2-4i\rho} \end{matrix} \right], \tag{3.5}$$

and $\psi_{-\rho}(y)$ *for* $\rho \in \mathbb{C}\backslash\frac{1}{2i}\mathbb{N}$ *are solutions of the equation*

$$\Delta_q^{(0)} f(y) = \lambda(\rho) f(y),$$

where

$$\lambda(\rho) = -\frac{(1-q^{1+2i\rho})(1-q^{1-2i\rho})}{(1-q^2)^2}.$$

Moreover,

$$\varphi_\rho(y) = \frac{\Gamma_{q^2}(2i\rho)}{\Gamma^2_{q^2}(\frac{1}{2}+i\rho)}\psi_\rho(y) + \frac{\Gamma_{q^2}(-2i\rho)}{\Gamma^2_{q^2}(\frac{1}{2}-i\rho)}\psi_{-\rho}(y). \tag{3.6}$$

Remark. $\varphi_\rho(y)$ is a q-analogue of the spherical function in the unit disc (see [5]).

Proposition 3.4 *[9, Corollary 4.2]. The spectrum of* $\Delta_q^{(0)}$ *is simple purely continuous and coincides with the segment*

$$\left[-\frac{1}{(1-q)^2}; -\frac{1}{(1+q)^2}\right].$$

Proposition 3.5 *[9, Proposition 4.17]. Consider the Borel measure* $d\sigma$ *on the segment* $[0; \frac{2\pi}{h}]$ *(*$h = \ln q^{-2}$*) given by*

$$d\sigma(\rho) = \frac{1}{4\pi}\frac{h}{1-q^2}\frac{\Gamma^2_{q^2}(\frac{1}{2}-i\rho)\Gamma^2_{q^2}(\frac{1}{2}+i\rho)}{\Gamma_{q^2}(-2i\rho)\Gamma_{q^2}(2i\rho)}d\rho. \tag{3.7}$$

Then the linear operator

$$f(y) \mapsto \hat{f}(\rho) \stackrel{\mathrm{def}}{=} \int_0^1 \varphi_\rho(y) f(y) y^{-2} d_{q^2}y, \tag{3.8}$$

defined on functions with finite supports inside $q^{2\mathbb{Z}_+}$ *is extendable by continuity to a unitary operator*

$$u : L^2(d\nu)_q^{(0)} \to L^2(d\sigma).$$

For all $f \in L^2(d\nu)_q^{(0)}$

$$u \cdot \Delta_q^{(0)} f = \lambda(\rho) u f, \tag{3.9}$$

and the inverse operator is

$$f(\rho) \mapsto \int_0^{2\pi/h} \varphi_\rho(y) f(\rho) d\sigma(\rho). \tag{3.10}$$

Remark. Formulae (3.8), (3.10) present a decomposing in eigenfunctions of the operator $\Delta_q^{(0)}$. The function $\hat{f}(\rho)$ is called the spherical transform of $f(y)$ while $f(y)$ is the inverse spherical transform for $\hat{f}(\rho)$.

Now let us turn to proving of Theorem 3.2

Lemma 3.6 *Let $g_m(y)$ stand for the function such that*

$$\Delta_q^{(0)m} g_m(y) = f_0. \tag{3.11}$$

Then

$$g_m(q^{2N}) = (-1)^m (1-q^2)^{2m} (q^2;q^2)_\infty q^N \sum_{k=0}^{\infty} \frac{q^{2Nk+2k}}{(q^2;q^2)_k}$$

$$\times \mathrm{Res}_{\tau=q} \left(\frac{\tau^{N+m-1}(1-\tau^2)(q^{2k+2}\tau^2;q^2)_\infty d\tau}{(\tau-q)^m (1-q\tau)^m (q^{2k+1}\tau;q^2)_\infty^2} \right). \tag{3.12}$$

Proof of the lemma. Applying the spherical transform to the both sides of (3.11), using (3.9) and equality $\varphi_\rho(1)=1$ we get

$$\lambda(\rho)^m \hat{g}_m(\rho) = 1-q^2 \tag{3.13}$$

and then

$$\hat{g}_m(\rho) = (-1)^m \frac{(1-q^2)^{2m+1}}{(1-q^{1+2i\rho})^m (1-q^{1-2i\rho})^m}. \tag{3.14}$$

Now to obtain $g_m(y)$ in an explicit form it is sufficient to apply the inverse spherical transform to the both sides of (3.14), i.e.,

$$g_m(y) = (-1)^m (1-q^2)^{2m+1} \int_0^{2\pi/h} \frac{\varphi_\rho(y)}{(1-q^{1+2i\rho})^m (1-q^{1-2i\rho})^m} d\sigma(\rho). \tag{3.15}$$

Next, to compute the integral in the right side of (3.15) we replace $\varphi_\rho(y)$ by its decomposition into sum of two items (cf.(3.6)):

$$g_m(y) = c_m \int_0^{2\pi/h} \frac{\psi_\rho(y)}{(1-q^{1+2i\rho})^m (1-q^{1-2i\rho})^m} \frac{\Gamma_{q^2}^2(\frac{1}{2}-i\rho)}{\Gamma_{q^2}(-2i\rho)} d\rho$$

$$+c_m \int_0^{2\pi/h} \frac{\psi_{-\rho}(y)}{(1-q^{1+2i\rho})^m (1-q^{1-2i\rho})^m} \frac{\Gamma_{q^2}^2(\frac{1}{2}+i\rho)}{\Gamma_{q^2}(2i\rho)} d\rho, \tag{3.16}$$

where $c_m = \frac{(-1)^m}{4\pi}(1-q^2)^{2m} h$.

The two integrals in the right side of (3.16) are equal to each other (to check this one should replace ρ by $-\rho$ in the former integral and observe that all the functions under the integrals are $\frac{2\pi}{h}$-periodic).

Hence

$$g_m(y) = \frac{(-1)^m}{2\pi}(1-q^2)^{2m} h \int_0^{2\pi/h} \frac{\psi_\rho(y)}{(1-q^{1+2i\rho})^m (1-q^{1-2i\rho})^m} \frac{\Gamma_{q^2}^2(\frac{1}{2}-i\rho)}{\Gamma_{q^2}(-2i\rho)} d\rho. \tag{3.17}$$

Now let us make use of the equalities

$$\psi_\rho(y) = y^{1/2-i\rho} \sum_{k=0}^{\infty} \frac{(q^{1-2i\rho}; q^2)_k^2}{(q^{2-4i\rho}; q^2)_k (q^2; q^2)_k} q^{2k} y^k \tag{3.18}$$

and

$$\frac{\Gamma_{q^2}^2(\frac{1}{2} - i\rho)}{\Gamma_{q^2}(-2i\rho)} = \frac{\frac{(q^2;q^2)_\infty^2}{(q^{1-2i\rho};q^2)_\infty^2}(1-q^2)^{1+2i\rho}}{\frac{(q^2;q^2)_\infty}{(q^{-4i\rho};q^2)_\infty}(1-q^2)^{1+2i\rho}} = (q^2;q^2)_\infty \frac{(q^{-4i\rho}; q^2)_\infty}{(q^{1-2i\rho}; q^2)_\infty^2} . \tag{3.19}$$

Thus

$$g_m(y) = \frac{(-1)^m h}{2\pi}(1-q^2)^{2m}(q^2;q^2)_\infty y^{1/2} \sum_{k=0}^{\infty} \frac{q^{2k} y^k}{(q^2;q^2)_k}$$

$$\times \int_0^{2\pi/h} \frac{1}{(1-q^{1+2i\rho})^m (1-q^{1-2i\rho})^m} \frac{(q^{1-2i\rho}; q^2)_k^2}{(q^{2-4i\rho}; q^2)_k} \frac{(q^{-4i\rho}; q^2)_\infty}{(q^{1-2i\rho}; q^2)_\infty^2} y^{-i\rho} d\rho$$

(we have exchanged summation over k and integration over $[0; \frac{2\pi}{h}]$ because of the uniform convergence of the series (3.18) for any fixed $y \in q^{2\mathbb{Z}_+}$).

Let $y = q^{2N}$. Remind that $q = e^{-h/2}$. Then

$$\int_0^{2\pi/h} \frac{1}{(1-q^{1+2i\rho})^m (1-q^{1-2i\rho})^m} \frac{(q^{1-2i\rho}; q^2)_k^2}{(q^{2-4i\rho}; q^2)_k} \frac{(q^{-4i\rho}; q^2)_\infty}{(q^{1-2i\rho}; q^2)_\infty^2} q^{-2Ni\rho} d\rho$$

$$= \int_0^{2\pi/h} \frac{1}{(1-qe^{-hi\rho})^m (1-qe^{hi\rho})^m} \frac{(qe^{hi\rho}; q^2)_k^2}{(q^2 e^{2hi\rho}; q^2)_k} \frac{(e^{2hi\rho}; q^2)_\infty}{(qe^{hi\rho}; q^2)_\infty^2} e^{hNi\rho} d\rho$$

$$= \frac{1}{hi} \int_{\mathbb{T}} \frac{1}{(1-\frac{q}{\tau})^m (1-q\tau)^m} \frac{(q\tau; q^2)_k^2}{(q^2\tau^2; q^2)_k} \frac{(\tau^2; q^2)_\infty}{(q\tau; q^2)_\infty^2} \tau^{N-1} d\tau$$

$$= \frac{1}{hi} \int_{\mathbb{T}} \frac{\tau^{N+m-1}}{(\tau-q)^m (1-q\tau)^m} (1-\tau^2) \frac{(q^{2k+2}\tau^2; q^2)_\infty}{(q^{2k+1}\tau; q^2)_\infty^2} d\tau .$$

This completes the proof of the lemma. □

Thus by Lemma 3.6 we have

$$g_1(q^{2N}) = -(1-q^2)^2 (q^2;q^2)_\infty q^{2N} \sum_{k=0}^{\infty} \frac{q^{2Nk+2k}}{(q^2;q^2)_k} \cdot \frac{(q^{2k+4}; q^2)_\infty}{(q^{2k+2}; q^2)_\infty^2}$$

$$= -(1-q^2)^2 (q^2;q^2)_\infty q^{2N} \sum_{k=0}^{\infty} \frac{q^{2Nk+2k}}{(q^2;q^2)_k (1-q^{2k+2})(q^{2k+2};q^2)_\infty}$$

$$= -(1-q^2)^2 (q^2;q^2)_\infty q^{2N} \sum_{k=0}^{\infty} \frac{q^{2Nk+2k}}{(1-q^{2k+2})(q^2;q^2)_\infty} .$$

Hence

$$g_1(y) = -(1-q^2)\sum_{m=1}^{\infty} \frac{q^{-2}-1}{q^{-2m}-1} y^m. \tag{3.20}$$

For g_2 the calculations are much more complicated

$$\mathrm{Res}_{\tau=q}\left(\frac{\tau^{N+1}(1-\tau^2)(q^{2k+2}\tau^2;q^2)_\infty d\tau}{(\tau-q)^2(1-q\tau)^2(q^{2k+1}\tau;q^2)_\infty^2}\right)$$

$$= \frac{d}{d\tau}\left(\frac{\tau^{N+1}(1-\tau^2)}{(1-q\tau)^2}\cdot(q^{2k+2}\tau^2;q^2)_\infty \frac{1}{(q^{2k+1}\tau;q^2)_\infty^2}\right)_{\tau=q}.$$

One proves that

$$\frac{d}{d\tau}(\tau;q^2)_\infty = -(\tau;q^2)_\infty \cdot L_\infty(\tau)$$

(here we use the notation L_∞ from Section 2). Thus

$$\frac{d}{d\tau}\left(\frac{\tau^{N+1}(1-\tau^2)}{(1-q\tau)^2}\cdot(q^{2k+2}\tau^2;q^2)_\infty \frac{1}{(q^{2k+1}\tau;q^2)_\infty^2}\right)$$

$$= \frac{(N+1)\tau^N-(N+3)\tau^{N+2}}{(1-q\tau)^2}\cdot\frac{(q^{2k+2}\tau^2;q^2)_\infty}{(q^{2k+1}\tau;q^2)_\infty^2}$$

$$+\frac{2q(\tau^{N+1}-\tau^{N+3})}{(1-q\tau)^3}\cdot\frac{(q^{2k+2}\tau^2;q^2)_\infty}{(q^{2k+1}\tau;q^2)_\infty^2}+\frac{\tau^{N+1}-\tau^{N+3}}{(1-q\tau)^2}$$

$$\times\left(-2q^{2k+2}\tau L_\infty(q^{2k+2}\tau^2)\frac{(q^{2k+2}\tau^2;q^2)_\infty}{(q^{2k+1}\tau;q^2)_\infty^2}+2q^{2k+1}L_\infty(q^{2k+1}\tau)\frac{(q^{2k+2}\tau^2;q^2)_\infty}{(q^{2k+1}\tau;q^2)_\infty^2}\right).$$

For $\tau = q$ we get

$$\frac{(N+1)q^N-(N+3)q^{N+2}}{(1-q^2)^2}\cdot\frac{(q^{2k+4};q^2)_\infty}{(q^{2k+2};q^2)_\infty^2}+\frac{2q(q^{N+1}-q^{N+3})}{(1-q^2)^3}\cdot\frac{(q^{2k+4};q^2)_\infty}{(q^{2k+2};q^2)_\infty^2}$$

$$-\frac{q^{N+1}-q^{N+3}}{(1-q^2)^2}\cdot 2q^{2k+3}L_\infty(q^{2k+4})\frac{(q^{2k+4};q^2)_\infty}{(q^{2k+2};q^2)_\infty^2}$$

$$+\frac{q^{N+1}-q^{N+3}}{(1-q^2)^2}2q^{2k+1}L_\infty(q^{2k+2})\frac{(q^{2k+4};q^2)_\infty}{(q^{2k+2};q^2)_\infty^2}$$

$$= q^N\frac{(N+1)-(N+3)q^2}{(1-q^2)^2}\cdot\frac{1}{(1-q^{2k+2})(q^{2k+2};q^2)_\infty}$$

$$+\frac{2q^{N+2}}{(1-q^2)^2}\cdot\frac{1}{(1-q^{2k+2})(q^{2k+2};q^2)_\infty}$$

$$-\frac{q^{N+1}}{(1-q^2)}\cdot 2q^{2k+3}L_\infty(q^{2k+4})\frac{1}{(1-q^{2k+2})(q^{2k+2};q^2)_\infty}$$

$$+\frac{q^{N+1}}{(1-q^2)}2q^{2k+1}L_\infty(q^{2k+2})\frac{1}{(1-q^{2k+2})(q^{2k+2};q^2)_\infty}$$

$$= \frac{1}{(1-q^2)(1-q^{2k+2})(q^{2k+2};q^2)_\infty}$$
$$\times \left(q^N(N+1) + 2q^{2k+1}\cdot q^{N+1}(L_\infty(q^{2k+2}) - q^2 L_\infty(q^{2k+4}))\right).$$

But

$$L_\infty(q^{2k+2}) - q^2 L_\infty(q^{2k+4}) = \sum_{m=0}^{\infty}\frac{q^{2m}}{1-q^{2m+2k+2}} - \sum_{m=0}^{\infty}\frac{q^{2m+2}}{1-q^{2m+2k+4}} = \frac{1}{1-q^{2k+2}}.$$

Hence finally we obtain

$$\mathrm{Res}_{\tau=q}\left(\frac{\tau^{N+1}(1-\tau^2)(q^{2k+2}\tau^2;q^2)_\infty d\tau}{(\tau-q)^2(1-q\tau)^2(q^{2k+1}\tau;q^2)^2_\infty}\right)$$
$$= \frac{1}{(1-q^2)(1-q^{2k+2})^2(q^{2k+2};q^2)_\infty}\cdot\left(q^N(N+1)(1-q^{2k-2}) + 2q^N\cdot q^{2k+2}\right),$$

and, using (3.12),

$$g_2(q^{2N}) = (1-q^2)^3(q^2;q^2)_\infty q^{2N}\sum_{k=0}^{\infty}\frac{q^{2Nk+2k}}{(q^2;q^2)_k(1-q^{2k+2})^2(q^{2k+2};q^2)_\infty}$$
$$\times((N+1)(1-q^{2k+2}) + 2q^{2k+2}).$$

Now to complete the proof of Theorem 3.2 it is sufficient to replace q^{2N} by y and N by $-\frac{1}{h}\ln y$ in the last formula.

Remark.

1. The explicit form (3.2) of $g_1(y)$ was obtained in [9, Proposition 1.1] in another way.
2. The function $g_2(y)$ given by (3.3) can be treated as a q-analogue of Rogers' dilogarithm.

4 Some more auxiliary results: quantum symmetry

Remind that the quantum universal enveloping algebra $U_q\mathfrak{sl}_2$ is a Hopf algebra over $\mathbb{C}$ determined by the generators K, K^{-1}, E, F and the relations

$$KK^{-1} = K^{-1}K = 1, \quad K^{\pm1}E = q^{\pm2}EK^{\pm1}, \quad K^{\pm1}F = q^{\mp2}FK^{\pm1},$$
$$EF - FE = (K-K^{-1})/(q-q^{-1}),$$
$$\Delta(K^{\pm1}) = K^{\pm1}\otimes K^{\pm1}, \quad \Delta(E) = E\otimes 1 + K\otimes E, \quad \Delta(F) = F\otimes K^{-1} + 1\otimes F.$$

Note that

$$\varepsilon(E) = \varepsilon(F) = \varepsilon(K^{\pm1}-1) = 0,$$
$$S(K^{\pm1}) = K^{\mp1}, \quad S(E) = -K^{-1}E, \quad S(F) = -FK,$$

with $\varepsilon: U_q\mathfrak{sl}_2 \to \mathbb{C}$ and $S: U_q\mathfrak{sl}_2 \to U_q\mathfrak{sl}_2$ being respectively the counit and the antipode of $U_q\mathfrak{sl}_2$.

Let F stand for an algebra over $\mathbb{C}$ with a unit and equipped also with a structure of $U_q\mathfrak{sl}_2$-module. F is called an $U_q\mathfrak{sl}_2$-module (covariant) algebra if

1. the multiplication $m: F\otimes F \to F$ is a morphism of $U_q\mathfrak{sl}_2$-modules;

2. for any $\xi \in U_q\mathfrak{sl}_2$
$$\xi(1) = \varepsilon(\xi) \cdot 1$$
(here 1 is the unit of F). Note that an element v of an $U_q\mathfrak{sl}_2$-module is called invariant if for any $\xi \in U_q\mathfrak{sl}_2$
$$\xi(v) = \varepsilon(\xi) \cdot v.$$

Let M be an $U_q\mathfrak{sl}_2$-module and F-bimodule for some covariant algebra F. Then M is called covariant if the multiplication maps
$$m_L : F \otimes M \to M, \quad m_R : M \otimes F \to M$$
are morphisms of $U_q\mathfrak{sl}_2$-modules.

Equip $U_q\mathfrak{sl}_2$ with the involution given by
$$E^* = -KF, \quad F^* = -EK^{-1}, \quad (K^{\pm 1})^* = K^{\pm 1}. \tag{4.1}$$
$U_q\mathfrak{su}_{1,1}$ is the $*$-Hopf algebra produced this way.

An involutive algebra F is said to be $U_q\mathfrak{su}_{1,1}$-module algebra (covariant $*$-algebra) if it is an $U_q\mathfrak{sl}_2$-module one and
$$(\xi f)^* = (S(\xi))^* \cdot f^*$$
for any $\xi \in U_q\mathfrak{su}_{1,1}$ and $f \in F$.

It is very well known (see, for instance, [7]) that $\mathrm{Pol}(\mathbb{C})_q$ can be equipped with a structure of a covariant $*$-algebra in the following way:
$$K^{\pm 1} z = q^{\pm 2} z, \quad Ez = -q^{1/2} z^2, \quad Fz = q^{1/2}, \tag{4.2}$$
$$K^{\pm 1} z^* = q^{\mp 2} z^*, \quad Ez^* = q^{-3/2}, \quad Fz = -q^{-5/2} z^{*2}. \tag{4.3}$$

The formulae (4.2),(4.3) imply: for any polynomial f
$$\begin{gathered} K^{\pm 1} f(y) = f(y), \quad Ef(y) = -\tfrac{q^{1/2}}{1-q^2} z(f(y) - f(q^2 y)), \\ Ff(y) = -\tfrac{q^{5/2}}{1-q^2}(f(y) - f(q^2 y)) z^*. \end{gathered} \tag{4.4}$$
(4.4) allow one to "transfer" the structure of $U_q\mathfrak{su}_{1,1}$-module from $\mathrm{Pol}(\mathbb{C})_q$ onto $D(\mathbb{U})_q$.

Remark. The functional (2.4) possesses the following properties: for any $f \in D(\mathbb{U})_q$, $\xi \in U_q\mathfrak{sl}_2$

1. $\int\limits_{\mathbb{U}_q} f^* d\nu = \int\limits_{\mathbb{U}_q} f d\nu$ (follows from the definition);
2. $\int\limits_{\mathbb{U}_q} f^* f d\nu > 0, f \neq 0$ ([7, Remark 3.6]);
3. $\int\limits_{\mathbb{U}_q} \xi f d\nu = \varepsilon(\xi) \cdot \int\limits_{\mathbb{U}_q} f d\nu$ ([7, Theorem 3.5]).

These properties allow one to regard the functional as a q-analogue of the $SU(1,1)$-invariant integral.

Note (see [7, Proposition 4.1]) that 1–3 imply: for any $f_1, f_2 \in D(\mathbb{U})_q$, $\xi \in U_q\mathfrak{sl}_2$
$$(\xi f_1, f_2) = (f_1, \xi^* f_2), \tag{4.5}$$

where $(f_1, f_2) \stackrel{\mathrm{def}}{=} \int\limits_{\mathbb{U}_q} f_2^* f_1 d\nu$.

The following formulae can be obtained

$$K^{\pm 1} z^j f(y) = q^{\pm 2j} z^j f(y), \quad K^{\pm 1} f(y) z^{*j} = q^{\mp 2j} f(y) z^{*j}, \tag{4.6}$$

$$E(z^j f(y)) = -\frac{q^{1/2}}{1-q^2} z^{j+1} (f(y) - q^{2j} f(q^2 y)), \tag{4.7}$$

$$E(f(y) z^{*j}) = -\frac{q^{1/2}}{1-q^2} ((y - q^{-2j}) f(y) + (1-y) f(q^{-2} y)) z^{*(j-1)}, \quad j \geq 1, \tag{4.8}$$

$$F(z^j f(y)) = -\frac{q^{5/2}}{1-q^2} z^{j-1} ((y - q^{-2j}) f(y) + (1-y) f(q^{-2} y)), \quad j \geq 1, \tag{4.9}$$

$$F(f(y) z^{*j}) = -\frac{q^{5/2}}{1-q^2} (f(y) - q^{2j} f(q^2 y)) z^{*(j+1)}. \tag{4.10}$$

Impose the notation $l_{i,j}$, $i = 0, 1, 2, \ldots$, $j = 0, \pm 1, \pm 2, \ldots$, for the functional

$$\sum_{m>0} z^m \psi_m(y) + \psi_0(y) + \sum_{m>0} \psi_{-m}(y) z^{*m} \mapsto \psi_j(q^{2i})$$

on the space $D(\mathbb{U})'_q$ (see Section 1). Endow $D(\mathbb{U})'_q$ with the weakest among the topologies in which all the linear functionals $l_{i,j}$ are continuous. Obviously, $D(\mathbb{U})_q$ is a dense subspace in $D(\mathbb{U})'_q$. As a straightforward consequence of (4.6)–(4.10) we get

Proposition 4.1 *Any element $\xi \in U_q\mathfrak{sl}_2$ defines a continuous linear operator $D(\mathbb{U})_q \to D(\mathbb{U})_q$ (here $D(\mathbb{U})_q$ is regarded as a topological vector space with the topology induced by the topology on $D(\mathbb{U})'_q$ described above).*

Corollary 4.2 *The $U_q\mathfrak{sl}_2$-action on $D(\mathbb{U})_q$ can be transferred by continuity onto the space $D(\mathbb{U})'_q$.*

In fact the $U_q\mathfrak{sl}_2$-module $D(\mathbb{U})'_q$ is a covariant $D(\mathbb{U})_q$-bimodule.

One can apply the above arguments to $D(\mathbb{U})_q \otimes D(\mathbb{U})_q$, $D(\mathbb{U} \times \mathbb{U})'_q$, $\{l_{i,j} \otimes l_{m,n}\}$ instead of $D(\mathbb{U})_q$, $D(\mathbb{U})'_q$, $\{l_{i,j}\}$ to make $D(\mathbb{U} \times \mathbb{U})'_q$ into a topological vector space and an $U_q\mathfrak{sl}_2$-module. The continuity of the $U_q\mathfrak{sl}_2$-action in $D(\mathbb{U} \times \mathbb{U})'_q$ may be proved just as in the case of $D(\mathbb{U})'_q$.

The results listed below are proved in [7],[8].

Proposition 4.3 *[7, Proposition 4.5]. An integral operator with a kernel K is a morphism of the $U_q\mathfrak{sl}_2$-module $D(\mathbb{U})_q$ onto the $U_q\mathfrak{sl}_2$-module $D(\mathbb{U})'_q$ iff K is an invariant.*

Proposition 4.4 *[8, Section 6]. G_l given by (2.12) is invariant for any $l \in \mathbb{C}$.*

Proposition 4.5 *[7, Theorem 4.3, Proposition 5.2]* Δ_q *being regarded as an operator* $D(\mathbb{U})_q \to D(\mathbb{U})'_q$ *is a morphism of* $U_q\mathfrak{sl}_2$*-modules. Moreover,* $\Delta_q = q^{-1}\Omega$ *where*

$$\Omega \stackrel{\text{def}}{=} FE + \frac{1}{(q^{-1}-q)^2}(q^{-1}K^{-1} + qK - q - q^{-1}) \tag{4.11}$$

is an element of the centre of $U_q\mathfrak{sl}_2$ *called the Casimir element.*

Proposition 4.6 *[7, Theorem 3.9].* f_0 *given by (3.1) generates the* $U_q\mathfrak{sl}_2$*-module* $D(\mathbb{U})_q$*.*

Corollary 4.7 *Let* A, B *be morphisms of* $U_q\mathfrak{sl}_2$*-modules* $D(\mathbb{U})_q \to D(\mathbb{U})'_q$*. Then* $A = B$ *iff* $Af_0 = Bf_0$*.*

5 Proof of Theorem 2.2: reduction to the results of Section 3 about radial part of the quantum Laplacian

Firstly it should be proved that the integral operators in the right-hand sides of (2.15) and (2.16) are well defined (i.e., $\mathbb{G}_1$ and $\mathbb{G}_2$ do belong the space $D(\mathbb{U}\times\mathbb{U})'_q$). It could be done just as in the case of G_l and $\hat{G}_N$ (see Section 1) and we don't adduce such calculations.

Lemma 5.1 *For any* $N \in \mathbb{N}$ *the kernel* $\hat{G}_N$ *given by (2.13) is invariant.*

Proof. In our case the invariance of $\hat{G}_N$ means

$$E(\hat{G}_N) = F(\hat{G}_N) = (K^{\pm 1} - 1)(\hat{G}_N)$$

and follows from the continuity of the $U_q\mathfrak{sl}_2$-action in $D(\mathbb{U}\times\mathbb{U})'_q$ and Proposition 4.4. □

Lemma 5.2

$$\int\limits_{\mathbb{U}_q} \mathbb{G}_1 f_0 d\nu = g_1(y), \qquad \int\limits_{\mathbb{U}_q} \mathbb{G}_2 f_0 d\nu = g_2(y), \tag{5.1}$$

where $g_1(y)$*,* $g_2(y)$ *are given by (3.2) and (3.3) respectively.*

Proof. (5.1) reduce to (2.12), (2.14) and the following equalities:

1. $z^* \cdot f_0 = 0$ (see [7, Proposition 3.1]);
2. $f(y) \cdot f_0 = f(1)$ (follows from the definition (3.1) of f_0);
3. $\int\limits_{\mathbb{U}_q} z^k f_0 d\nu = \begin{cases} 1-q^2, & k=0, \\ 0, & k=1,2,\ldots \end{cases}$ (follows from the definition (2.4) of the integral).

□

Thus we have proved (Proposition 4.4, Lemma 5.1) that the operators in the right-hand sides of (2.15) and (2.16) are morphisms of $U_q\mathfrak{sl}_2$-modules. By Lemma 5.2 and Theorem 3.2

$$\int\limits_{\mathbb{U}_q} \mathbb{G}_1 f_0 d\nu = \Delta_q^{-1} f_0,$$

$$\int_{\mathbb{U}_q} \mathbb{G}_2 f_0 d\nu = \Delta_q^{-2} f_0.$$

By Corollary 4.7 to complete the proof of Theorem 2.2 it suffices now to prove the following lemma.

Lemma 5.3 *Δ_q^{-1} and Δ_q^{-2} being regarded as operators from $D(\mathbb{U})_q$ onto $D(\mathbb{U})_q'$ are morphisms of $U_q\mathfrak{sl}_2$-modules.*

Proof. Let t_ϕ be the automorphism of the algebra $\mathrm{Pol}(\mathbb{C})_q$ given by

$$t_\phi(z) = e^{i\phi}z, \quad t_\phi(z^*) = e^{-i\phi}z^*.$$

Impose the same notation t_ϕ for the automorphism

$$\sum_{m>0} z^m \psi_m(y) + \psi_0(y) + \sum_{m>0} \psi_{-m}(y) z^{*m}$$

$$\mapsto \sum_{m>0} e^{im\phi} z^m \psi_m(y) + \psi_0(y) + \sum_{m>0} e^{-im\phi} \psi_{-m}(y) z^{*m}$$

of the algebra $D(\mathbb{U})_q$.

Obviously, each operator t_ϕ can be extend to a unitary operator $L^2(d\nu)_q \to L^2(d\nu)_q$ and thus we obtain a unitary representation of the group $U(1)$.

Let's prove that for any ϕ

$$t_\phi \cdot \Delta_q = \Delta_q \cdot t_\phi.$$

Indeed, let L_n be the subspace in $D(\mathbb{U})_q$ of function of the form $z^n f(y)$ (for $n > 0$), $f(y)$ (for $n = 0$) or $f(y)z^{*n}$ (for $n < 0$). Then it is obvious that

$$L_n = \{f \in D(\mathbb{U})_q : t_\phi(f) = e^{in\phi} f, \phi \in [0; 2\pi)\} = \{f \in D(\mathbb{U})_q : K(f) = q^{2n} f\}.$$

The latter equality and Proposition 4.5 imply $\Delta_q(L_n) \subset L_n$ and thus

$$t_\phi(\Delta_q(f)) = \Delta_q(t_\phi(f))$$

for any t_ϕ and $f \in D(\mathbb{U})_q$.

Let $\overline{L_n}$ be the closure of L_n in $L^2(d\nu)_q$. It is evident that $\Delta_q(\overline{L_n}) \subset \overline{L_n}$ and, moreover, $\Delta_q^{-1}(\overline{L_n}) \subset \overline{L_n}$ (this follows from the invertibility of Δ_q).

Denote by $L^2(d\nu)_q^{fin}$ the space $\bigoplus_{n \in \mathbb{Z}} \overline{L_n}$ (note that $\overline{L_n} \perp \overline{L_m}$ for $n \neq m$). We have established that

$$\Delta_q(L^2(d\nu)_q^{fin}) \subset L^2(d\nu)_q^{fin},$$

and

$$\Delta_q^{-1}(L^2(d\nu)_q^{fin}) \subset L^2(d\nu)_q^{fin}.$$

Obviously, $D(\mathbb{U})_q \subset L^2(d\nu)_q^{fin}$ and therefore

$$\Delta_q^{-m}(D(\mathbb{U})_q) \subset L^2(d\nu)_q^{fin}$$

for any $m \in \mathbb{N}$.

To complete the proof of Lemma 5.3 it suffices to prove that $L^2(d\nu)_q^{fin}$ is an $U_q\mathfrak{sl}_2$-submodule in $D(\mathbb{U})'_q$. Thus we have to verify inclusions

$$K(L^2(d\nu)_q^{fin}) \subset L^2(d\nu)_q^{fin}, \tag{5.2}$$

$$E(L^2(d\nu)_q^{fin}) \subset L^2(d\nu)_q^{fin}, \tag{5.3}$$

$$F(L^2(d\nu)_q^{fin}) \subset L^2(d\nu)_q^{fin}. \tag{5.4}$$

(5.2) is evident. Let us prove (5.3) ((5.4) can be proved in a similar way).

Formulae (4.8), (4.10), imply $E(L_n) \subset L_{n+1}$ and we need to prove that E is extendable onto $\overline{L_n}$. Let $f \in \overline{L_n}$. Then (see (4.5))

$$\begin{aligned}
(Ef, Ef) = (f, E^*Ef) &\overset{(\text{see}(4.1))}{=} -(f, KFEf) \\
&\overset{(\text{see}(4.11))}{=} -(f, K \cdot (\Omega - \frac{q^{-1}K^{-1} + qK - q^{-1} - q}{(q^{-1} - q)^2})f) \\
&= -(f, K\Omega f) + (f, \frac{q^{-1} + qK^2 - (q^{-1} + q)K}{(q^{-1} - q)^2} f) \\
&= -q^{2n}(f, \Omega f) + \frac{q^{-1} + q^{4n+1} - (q^{-1} + q)q^{2n}}{(q^{-1} - q)^2}(f, f) \\
&\overset{(\text{Proposition 4.5})}{=} -q^{2n+1}(f, \Delta_q f) + \frac{q^{-1} + q^{4n+1} - (q^{-1} + q)q^{2n}}{(q^{-1} - q)^2}(f, f).
\end{aligned}$$

So the boundedness of Δ_q allows one to establish the boundedness of $E : L_n \to L_{n+1}$. This completes the proof of Lemma 5.3 and thus of Theorem 2.2.

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